

# Integral and derivative dispersion relations in the analysis of the data on $pp$ and $\bar{p}p$ forward scattering.

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## Abstract

Integral and derivative dispersion relations (DR) are considered for the  $pp$  and  $\bar{p}p$  forward scattering amplitudes. A new representation for the derivative DR, valid at lower energies than the standard one, is obtained. The data on the total cross sections of  $pp(\bar{p}p)$  interaction as well as those on the ratio of the real part to the imaginary part of the forward amplitude are analyzed within various forms of the DR and high-energy Regge models. It is shown that three models for the pomeron, simple pole with intercept larger than one, triple pole pomeron and double pole pomeron (both with intercept equal to one) lead to practically equivalent descriptions of the data at  $\sqrt{s} > 5$  GeV. It is also shown that the low-energy part of the dispersion integral (from the two-proton threshold up to  $\sqrt{s} = 5$  GeV) allows one to reproduce well the data on  $\rho$  at lower energies without additional free parameters.

## 1 Introduction

The energy dependence of the hadronic total cross sections as well as that of the parameters  $\rho = \Re A(s, 0)/\Im A(s, 0)$  - the ratios of the real to the imaginary part of the forward scattering amplitudes - was widely discussed quite a long time ago (see [1, 2, 3, 4] and references therein). However, in spite of recent detailed investigations on the subject, the theoretical situation remains somewhat undecided, mainly because of the  $\rho$  parameter.

In the papers [5], all available data on  $\sigma_{tot}(s)$  and  $\rho(s)$  for hadron-hadron, photon-hadron and photon-photon interactions were considered. Many analytical models for the forward scattering amplitudes were fitted and compared. The ratio  $\rho$  was calculated in explicit form, from the imaginary part parametrised explicitly by contributions from the pomeron and secondary reggeons. The values of the free parameters were determined from the fit to the data at  $s \geq s_{min}$ , where  $\sqrt{s_{min}} = 5$  GeV. Omitting all details, we note here the main two conclusions. The best description of the data (with the minimal  $\chi^2/dof$ , where  $dof$  is the number of degrees of freedom) is obtained for the model with  $\sigma_{tot}$  rising as  $\log^2 s$ . The model

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with  $\sigma_{tot}(s) \propto s^\epsilon, \epsilon > 0$  was excluded from the list of the best models (in accordance with COMPETE criteria, see details in [5]).

Analysis of these results shows that they are due to a poor description of  $\rho$  data at low energy. On the other hand, there are a few questions concerning the explicit Regge-type models usually used for analysis and description of the data. How low in energy can the Regge parametrisations be extended, as they are written as functions of the asymptotic variable  $s$  rather than the "Regge" variable  $\cos \theta_t$  ( $= E/m$  in the laboratory system for identical colliding particles)? At which energies can the "asymptotic" normalization

$$\sigma_{tot}(s) = \frac{1}{s} \Im m A(s, 0) \quad (1)$$

instead of the standard one

$$\sigma_{tot}(s) = \frac{1}{2mE} \Im m A(s, 0) \quad (2)$$

be used? And last, how much do the analytic expressions for  $\rho$  based on the derivative dispersion relations deviate from those calculated in the integral form?

In this paper, we try to answer these questions considering three pomeron models (simple pole in the complex- $j$  plane with intercept above unity, double pole and triple pole with intercepts equal to unity) for  $pp$  and  $\bar{p}p$  interactions at  $\sqrt{s} \geq 5$  GeV.

## 2 Integral and derivative dispersion relations.

The amplitude  $A(s, t, u)$  is an analytic function of its variables. Consequently it must satisfy the integral dispersion relation (IDR) which can be derived from the Cauchy theorem on analytic functions. Generally, for proton-proton and proton-antiproton amplitudes, two subtractions must be made. However, assuming, in accordance with many analyses, that the odderon does not contribute asymptotically, one can show that the dispersion relations for  $pp$  and  $\bar{p}p$  amplitudes can be reduced to those with one subtraction constant [1]:

$$\rho_{\pm} \sigma_{\pm} = \frac{B}{2m_p p} + \frac{E}{\pi p} P \int_{m_p}^{\infty} \left[ \frac{\sigma_{\pm}}{E'(E' - E)} - \frac{\sigma_{\mp}}{E'(E' + E)} \right] p' dE' \quad (3)$$

where  $m_p$  is the proton mass,  $E$  and  $p$  are the energy and momentum of the proton in the laboratory system, and  $B$  is a subtraction constant, usually determined from the fit to the data. The indices  $+$ ( $-$ ) stand respectively for the  $pp$  and ( $\bar{p}p$ ) amplitudes. The standard normalization (2) is chosen in Eq.(3).

In the above expression, the pole contributions and the part of the integral over the unphysical cut from the two-pion to the two-proton threshold are omitted because they are  $\lesssim 1\%$  (see, e.g. [6]) in the region of interest ( $\sqrt{s} \geq 5$  GeV). In other words, we shall investigate here only the contribution of the physical region to the dispersion integral.

The derivative dispersion relations (DDR) were obtained [2, 3] separately for crossing-even and crossing-odd amplitudes

$$f_{\pm}(s, 0) = A_{+}(s, 0) \pm A_{-}(s, 0). \quad (4)$$

They are very useful in a practice due to their simple analytical form at high energies,  $E \gg m_p$ :

$$\Re f_+(E, 0) \approx (E/m_p)^\alpha \tan \left[ \frac{\pi}{2} \left( \alpha - 1 + E \frac{d}{dE} \right) \right] \Im m f_+(E, 0) / (E/m_p)^\alpha. \quad (5)$$

However it is important to estimate the corrections to these asymptotic relations (5) if one is to use them at finite  $s$ .

The starting point is the dispersion integral relation for an even amplitude  $f_+$  (with poles and subtraction constant omitted – they will be re-introduced at the end of calculations –). The amplitude is normalized so that  $2m_p p_{lab} \sigma_t^+ = \Im m f_+(E, 0)$ . Let us consider the relation at  $t = 0$ :

$$\Re f_+(E, 0) = \frac{2E^2}{\pi} \text{P} \int_{m_p}^{\infty} \frac{dE'}{E'(E'^2 - E^2)} \Im m f_+(E', 0) \quad (6)$$

and represent it in the form

$$\frac{2E^2}{\pi} \text{P} \int_{m_p}^{\infty} \frac{dE' (E'/m_p)^\alpha}{E'(E'^2 - E^2)} g(E'),$$

where  $g(E') = \Im m f_+(E', 0) / (E'/m_p)^\alpha$  and  $\alpha$  is an arbitrary constant (it is not an intercept!).

Integrating by parts (and using the notations  $F(E') = (E'/m_p)^{\alpha-1} g(E')$ ,  $F'(E) \equiv \frac{dF}{dE}$  for all functions), we obtain, assuming that  $\Im m f_+(m_p, 0) = 0$ :

$$\Re f_+(E, 0) = \frac{E}{\pi m_p} \int_{m_p}^{\infty} dE' \ln \left( \frac{E' + E}{|E' - E|} \right) F'(E').$$

Now let us transform the integrand functions to the forms

$$F'(E') = \frac{d}{dE'} \left[ (E'/m_p)^{\alpha-1} \frac{\Im m f_+(E', 0)}{(E'/m_p)^\alpha} \right] = \frac{1}{m_p} (E'/m_p)^{\alpha-2} \left( \alpha - 1 + \frac{d}{d\xi'} \right) g(\xi'),$$

$$\ln \frac{E' + E}{|E' - E|} = \ln \frac{e^{\xi'} + e^\xi}{|e^{\xi'} - e^\xi|} = \ln \frac{e^{(\xi' - \xi)/2} + e^{-(\xi' - \xi)/2}}{|e^{(\xi' - \xi)/2} - e^{-(\xi' - \xi)/2}|} = \ln \left| \coth \frac{1}{2}(\xi' - \xi) \right|,$$

where  $\xi' = \ln(E'/m_p)$  and  $\xi = \ln(E/m_p)$ .

Taking into account the above expression, one can write the integral (6) in the form

$$\begin{aligned} \Re f_+(E, 0) &= \frac{E}{\pi m_p} \left\{ \ln \left( \frac{E+m_p}{E-m_p} \right) \Im m f_+(m_p, 0) + \right. \\ &\quad \left. + \int_0^\infty d\xi' e^{(\alpha-1)\xi'} \ln \left( \left| \coth \frac{1}{2}(\xi' - \xi) \right| \right) \left( \alpha - 1 + \frac{d}{d\xi'} \right) g(\xi') \right\}. \end{aligned} \quad (7)$$

Rewriting the integrand logarithm (at  $x = \xi' - \xi \neq 0$ ) in (7)

$$\ln \left| \coth \frac{1}{2}x \right| = \begin{cases} \ln \frac{1+e^{-x}}{1-e^{-x}}, & x > 0, \\ \ln \frac{1+e^x}{1-e^x}, & x < 0 \end{cases} = \ln \frac{1+e^{-|x|}}{1-e^{-|x|}} = 2 \sum_{p=0}^{\infty} \frac{e^{-(2p+1)|x|}}{2p+1}.$$

and expanding other factors in the integrand of (7) in powers of  $\xi' - \xi$  (it does not depend on  $\xi$ , this is just a trick allowing explicitly to perform the integration and to present the result as a series)<sup>4</sup>

$$\left(\alpha - 1 + \frac{d}{d\xi'}\right) g(\xi') \equiv \tilde{g}(\xi') = \sum_{k=0}^{\infty} \frac{(\xi' - \xi)^k}{k!} D_{\xi}^k \tilde{g}(\xi),$$

where  $D_{\xi} = \frac{d}{d\xi}$  and  $\xi \neq 0$ ,

$$e^{(\alpha-1)\xi'} = e^{(\alpha-1)\xi} e^{(\alpha-1)(\xi' - \xi)} = e^{(\alpha-1)\xi} \sum_{n=0}^{\infty} \frac{(\alpha-1)^n}{n!} (\xi' - \xi)^n,$$

one can write

$$\Re f_+(E, 0) = \frac{2E}{\pi m_p} e^{(\alpha-1)\xi} \sum_{p=0}^{\infty} \frac{1}{2p+1} \sum_{k=0}^{\infty} \sum_{n=0}^{\infty} \frac{(\alpha-1)^n}{k!n!} \cdot \mathcal{I}(\xi; p, k, n) \cdot D_{\xi}^k \tilde{g}(\xi)$$

where

$$\begin{aligned} \mathcal{I}(\xi; p, k, n) &= \int_0^{\infty} d\xi' e^{-(2p+1)|\xi' - \xi|} (\xi' - \xi)^{k+n} \\ &= \frac{1}{(2p+1)^{k+n+1}} [\Gamma(k+n+1) + (-1)^{k+n} \gamma(k+n+1, \xi)(2p+1)] \end{aligned} \quad (8)$$

and  $\gamma(a, x)$  is the incomplete gamma function. We should note that one can prove that the asymptotic expression for DDR as well as the corrections to them do not depend on the auxiliary parameter  $\alpha$  (thus it is unreasonable to determine the parameter  $\alpha$  from a fit to data). So we may put, for convenience,  $\alpha = 1$ . The first term in (8) gives the asymptotic form (5) [2, 3, 4] written at  $\alpha = 1$ .

After some transformations we obtain for the corrections the series expansion in powers of  $m_p/E$

$$\Re f_+^{(cor)}(E, 0) = -\frac{2E}{\pi m_p} \sum_{p=0}^{\infty} \frac{C_+(p)}{2p+1} e^{-\xi(2p+1)}. \quad (9)$$

where

$$C_+(p) = \frac{e^{-\xi D_{\xi}}}{2p+1 + D_{\xi}} [\Im m f_+(E, 0) - E \Im m f'_+(E, 0)].$$

It should be noted that despite an apparent dependence on  $\xi = \ln(E/m_p)$ , the above expression for  $C_+(p)$  in fact does not depend on  $E$ . This can be proven using the properties of  $\exp(-\xi D_{\xi})$ .

Then, collecting all terms and adding a subtraction constant  $B_+$ , we obtain the final expression for  $\Re f_+(E, 0)$

$$\Re f_+(E, 0) = B_+ + E \tan \left[ \frac{\pi}{2} E \frac{d}{dE} \right] \frac{\Im m f_+(E, 0)}{E} - \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{C_+(p)}{2p+1} \left( \frac{m_p}{E} \right)^{2p}. \quad (10)$$

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<sup>4</sup>Note that it is not allowed if  $\xi \rightarrow 0$ .

A similar expression is obtained for the crossing-odd part of the amplitudes. If a contribution of asymptotic odderon is absent then

$$\Re f_-(E, 0) = -E \cot \left[ \frac{\pi}{2} \frac{Ed}{dE} \right] \frac{\Im m f_-(E, 0)}{E} - \frac{2}{\pi} \sum_{p=0}^{\infty} \frac{C_-(p)}{2p+1} \left( \frac{m_p}{E} \right)^{2p+1}, \quad (11)$$

where

$$C_-(p) = e^{-\xi D_\xi} \left\{ \frac{1}{2p + D_\xi} \Im m f'_-(E, 0) \right\}.$$

### 3 Phenomenology.

Our aim is to compare the fits of three pomeron models with  $\rho(s)$  calculated by two methods. The first one is the integral dispersion relation, and the second one is the asymptotic form of the derivative dispersion relation with a subtraction constant. We would like evaluate the importance of the normalization on the quality of the descriptions and on the values of the model parameters.

There are a few difficulties in an analysis of the available experimental data when the integral dispersion relations are applied.

- The imaginary part of the amplitude under consideration must be known in the whole kinematic region for a given process, starting from threshold.
- It is evident that the known parameterization of the cross sections, valid at  $s \geq s_{min}$ , cannot be used for the part of dispersion integral from the threshold  $s_{th}$  to  $s_{min}$ .
- The available experimental data on the parameters  $\rho$  for all processes,  $pp, \bar{p}p, \pi^\pm p$  and  $K^\pm p$  are of poor quality compared to those for total cross sections.

In this paper, we apply as a first step the IDR and DDR only to  $pp$  and  $\bar{p}p$  data. We have fitted the high-energy models to the data at  $\sqrt{s} \geq 5$  GeV.

#### 3.1 Low-energy data.

Low-energy total cross sections for  $pp$  (143 points) and  $\bar{p}p$  (220 points) interactions at  $\sqrt{s} < 5$  GeV were parameterized as functions of the proton momentum in the laboratory system by different expressions for different intervals of momentum.

The values of the free parameters are determined from a fit to the data [8] at  $s < s_{min}$  with the only constraint that the cross sections calculated from a fit to the data at  $s < s_{min}$  must coincide at  $s = s_{min}$  with those given by the fit to a specific high-energy model, valid for  $s > s_{min}$  (all data on  $\sigma$  and  $\rho$  are taken from [8]).

Thus we perform an overall fit in three steps.

1. The chosen model for high-energy cross-sections is fitted to the data on the cross sections only (without  $\rho$  data) at  $s > s_{min}$ .

2. The obtained "high-energy" parameters are fixed. The "low-energy" parameters are determined from the fit at  $s < s_{min}$ , but with  $\sigma_{pp}^{\bar{p}p}(s_{min})$  given by the first step.
3. The subtraction constant  $B_+$  is determined from the fit at  $s > s_{min}$  with all other parameters kept fixed.

Then, without fitting, we calculate the ratios  $\rho_{pp}$  and  $\rho_{\bar{p}p}$  at all energies above the physical threshold. The results of such procedure (for the cross sections) are given in Fig.1 and Fig.2, and will be detailed in a forthcoming paper.

### 3.2 High energy. Pomeron models.

As an example, we consider three models leading to different asymptotic behaviors for the total cross sections. For each model, we investigate how the integral and derivative dispersion relations work. We start from the explicit parameterization of the total  $pp$  and  $\bar{p}p$  cross-sections, then, to find the ratios of the real to imaginary parts, we apply the IDR making use of the second method for a calculation of the low-energy part of the dispersion integral. Then we compare results for ratios calculated through the DDR.

All the above-mentioned models include the contributions of pomeron,  $f$  and  $\omega$  reggeons (we consider these reggeons as effective ones because it is not reasonable to add other secondary reggeons provided only the  $pp$  and  $\bar{p}p$  data are fitted.)

$$\sigma_{pp}^{\bar{p}p} = \mathcal{P}(E) + R_f(E) \pm R_\omega(E), \quad (12)$$

where  $R_f = R_+$ ,  $R_\omega = R_-$  and

$$R_\pm(E) = g_\pm \left( \frac{E}{\lambda m_p} \right)^{\alpha_\pm(0)-1}. \quad (13)$$

The parameter  $\lambda$  can play a role only for the pomeron term in the triple pole model (see below), in the simple pole and dipole models  $\lambda = 1$  but we keep it to have a common form and notation for the three models.

When the imaginary part of amplitude is integrated in IDR we consider (for comparison) two kinds of normalization: the standard one defined in Eq. (2) and the asymptotic one given by Eq. (1). For the latter case, in the expressions for cross-sections, the replacement  $E/\lambda m_p \rightarrow s/s_1$  is made.

Besides, we compare our results with the models which are written as functions of  $-is$  and with the asymptotic normalization (1). We denote such fits as " $-is$ " fits.

#### 3.2.1 Simple pole pomeron model (SP).

In this model, the intercept of the pomeron trajectory is larger than unity [7]

$$\mathcal{P}(E) = g \left( \frac{E}{\lambda m_p} \right)^{\alpha_P(0)-1}. \quad (14)$$

We present the results of the fits using derivative dispersion relations (with a subtraction constant) and of standard fits with amplitudes defined in accordance with "– $is$ " rule:

$$\frac{1}{s}A_{pp}^{\bar{p}p}(s, 0) = ig(-is/s_1)^{\alpha_{\mathcal{P}}(0)-1} + ig_+(-is/s_1)^{\alpha_+(0)-1} \pm g_-(-is/s_1)^{\alpha_-(0)-1}. \quad (15)$$

### 3.2.2 Dipole pomeron model (DP).

The pomeron in this model is a double pole in the complex angular momentum plane with intercept  $\alpha_{\mathcal{P}}(0) = 1$ .

$$\mathcal{P}(E) = g_1 + g_2 \ln(E/\lambda m_p). \quad (16)$$

For the "– $is$ " fit we use

$$\frac{1}{s}A_{pp}^{\bar{p}p}(s, 0) = ig_1 + ig_2 \ln(-is/s_1) + ig_+(-is/s_1)^{\alpha_+(0)-1} \pm g_-(-is/s_1)^{\alpha_-(0)-1}. \quad (17)$$

### 3.2.3 Tripole pomeron model (TP)

The dominant term at high energy in this model is the contribution of the hardest  $j$ -plane Regge singularity allowed by unitarity: the triple pole at  $t = 0$  and  $j = 1$ . The form of this contribution is

$$\mathcal{P}(E) = g_1 + g_2 \ln^2(E/\lambda m_p). \quad (18)$$

The prescription "– $is$ " gives

$$\frac{1}{s}A_{pp}^{\bar{p}p}(s, 0) = ig_1 + ig_2 \ln^2(-is/s_1) + ig_+(-is/s_1)^{\alpha_+(0)-1} \pm g_-(-is/s_1)^{\alpha_-(0)-1}. \quad (19)$$

## 4 Results, discussion and conclusions.

Omitting many details of the fits, as well as the numerical values of the parameters, and postponing them for a forthcoming paper, we concentrate now on the main results and conclusions. The quality of the fits is presented in the Table, where  $\chi^2/N_p$  ( $N_p$  is the number of experimental points). In Figs. 3 and 4, we show the curves only for the simple-pole pomeron model. For other models the curves are practically the same in the region where data are available.

Table 1: The values of  $\chi^2$  obtained in the various pomeron models and through the different methods for the calculation of the ratio  $\rho$ .

	Standard normalization		High-energy normalization	
	IDR	DDR	IDR	"– $is$ ", $B = 0$
Simple Pole	1.0462	1.0646	1.1120	1.1209
Double Pole	1.0532	1.0454	1.1319	1.0793
Triple Pole	1.0438	1.0456	1.1114	1.1153

As one can see from the Table, all models give good descriptions of the data on  $\sigma_{tot}$  and  $\rho$ . Evidently, the fit with the integral dispersion relations and with the standard normalization is preferable. While the data on  $\sigma$  are described with  $\chi^2/dof \approx 0.91$ , the data on  $\rho$  are described less well, with a  $\chi^2/N_p \approx 1.5$ . However we think that this occurs because of the bad quality of the  $\rho$  data.

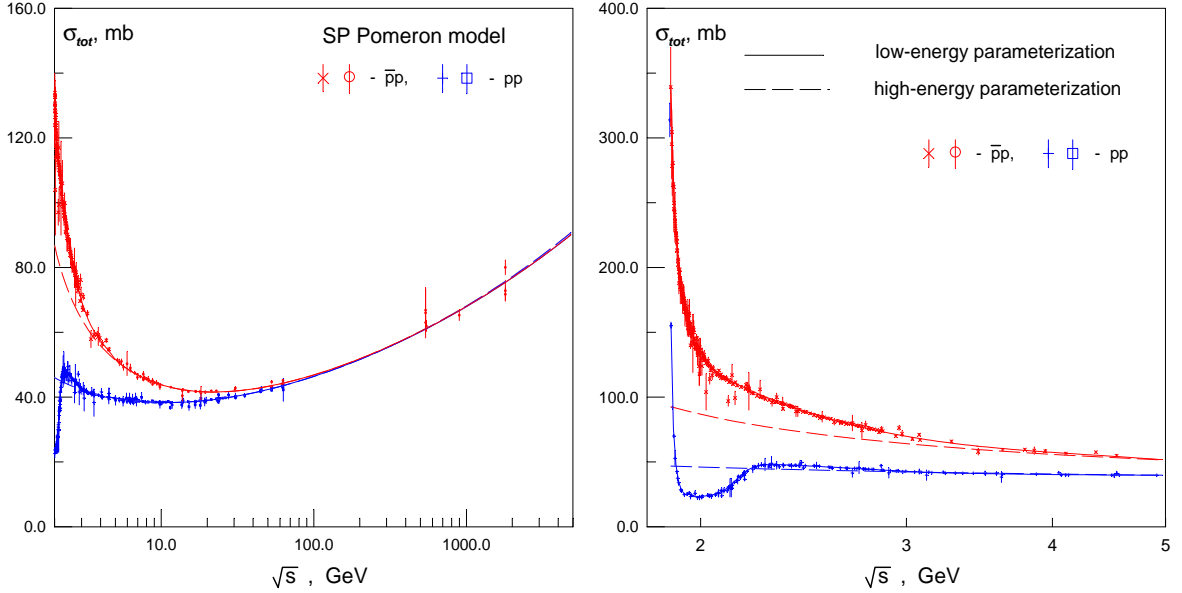


Figure 1: Cross sections in the SP pomeron model at all energies (left) and at low energies (right). The solid lines are the cross-sections tuned to SP model as described in the text. The dashed lines are the continuation of the high-energy SP parameterization to lower energies.

We would like to insist on the fact that the values of  $\rho$  calculated using DDR deviate from those calculated with IDR even at  $\sqrt{s} \lesssim 7 - 8$  GeV (see Figs. 3, 4). It means that in order to have more correct values of the  $\rho$  at such energies, one must use the IDR rather than explicit analytical expressions from the DDR.

The neglect of the subtraction constant, together with the use of asymptotic formulae in a non-asymptotic domain, may be the source of the conclusion of [5] excluding the simple-pole model from the list of the best models. Inclusion of these (non-asymptotic) terms improves the description of  $\rho$  considerably, and may lead to different conclusions regarding the simple-pole model.

However, in order to have these final conclusions, one will have to make a complete (a la COMPETE) analysis of the all data, including cross sections and  $\rho$  for  $\pi p$  and  $Kp$  interactions.



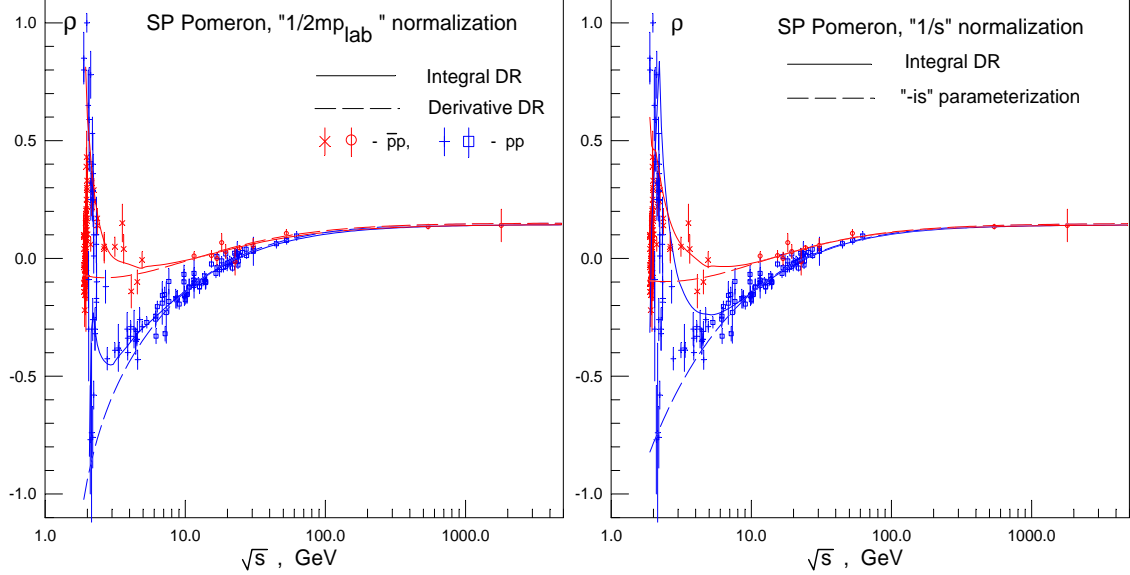


Figure 2: The ratios  $\rho$  in the SP model. Left: the integral and derivative dispersion relation fit with the standard normalization. Right: the IDR fit with asymptotic normalization and “ $-is$ ”-fit (see text for details).

## References

- [1] M.L. Goldberger, Y. Nambu, R. Oehme, *Ann. Phys.* **2**, 226 (1957);  
P. Söding, *Phys. Lett.* **8**, 285, (1964);  
M.M. Block, R.N. Cahn, *Rev. Mod. Phys.* **57**, 563 (1985);  
U. Amaldi *et al.*, *Phys. Lett.* **B66**, 390 (1985);  
P. Kroll, W. Schwenger, *Nucl. Phys.* **A503**, 865 (1989);  
C. Aguer *et al.* (UA4 Collab.), *Phys. Lett.* **B315**, 503, (1993);  
P. Desgrolard, M. Giffon, E.S. Martynov, *Nuovo Cimento*, **110A**, 537 (1997).
- [2] V.N. Gribov, A.A. Migdal, *Yad. Fiz.* **8**, 1002 (1968).
- [3] J.B. Bronzan, G.L. Kane, U.P. Sukhatme, *Phys. Lett.* **B49**, 272 (1974).
- [4] K. Kang, B. Nicolescu, *Phys. Rev.* **D11**, 2561 (1975).
- [5] J.R. Cudell *et al.* (COMPETE Collab.), *Phys. Rev.* **D65**, 074024 (2002);  
J.R. Cudell *et al.* (COMPETE Collab.), *Phys. Rev. Lett.* **89**, 201801 (2002),
- [6] A.I. Lengyel, V.I. Lengyel, *Yad. Fiz.* **11**, 669 (1970).
- [7] A. Donnachie, P.V. Landshoff, *Phys. Lett.* **B296**, 227 (1992).
- [8] <http://wwwppds.ihep.su:8001/hadron.html>